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SPECIFIC FEATURES OF THE APPLICATION OF ROUTH'S THEORY TO SYSTEMS WITH DIFFERENTIAL CONSTRAINTS†

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Specific points connected with the application of Routh's theory to the study of steady motions in systems with differential constraints are discussed, on the assumption that first integrals exist. The need to eliminate all dependent variables initially from the expressions for the first integrals is pointed out. The general conclusions are illustrated by investigating, as an example, the steady motions of a dynamically symmetric sphere on an a absolutely rough horizontal plane.

According to Routh's theory [1–9], if one of the first integrals of a system has critical (extremal) values, while the values of the other integrals remain fixed, this corresponds to (stable) real motions of the system, known as steady motions. It is assumed throughout that the equations of motion of the system may be written in the form

$$\dot{x} = f(x) \quad (x \in R^n,\ f(x) \in C^1 : R^n \to R^n) \tag{1}$$

while the first integrals have the form

$$U(x) = c = \text{const}\left(c \in R^m,\ U(x) \in C^2 : R^n \to R^m\right) \tag{2}$$

Since the motion of a system with differential constraints is not infrequently described by equations containing the reactions to the constraints or undetermined Lagrange multipliers, the application of Routh's theory to such systems requires special care. The point is that the above-mentioned equations of systems with differential constraints cannot be written in the form (1), since there are no corresponding differential equations for the reactions of the constraints or the undetermined Lagrange multipliers. To apply Routh's theory, therefore, one must first eliminate the dependent velocities from the expressions for all first integrals of the equations of motion of the system. The functions thus obtained will be first integrals of the Chaplygin, Voronets, Boltzmann–Hamel, etc. versions of the equations of motion, which do not contain the reactions to the constraints or undetermined Lagrange multipliers; such equations may be written in the form (1) and their first integrals take the form (2).

To illustrate these conclusions, let us consider as an example the steady motions of a dynamically symmetric sphere on an absolutely rough horizontal plane.

Let m be the mass of the sphere, I_1 and I_3 its equatorial and axial central moments of inertia, r its radius, a the distance from the sphere's centre of mass to its geometric centre, and g the acceleration due to gravity. Denote the velocity of the sphere's centre of mass and its angular velocity about its centre of mass by $\mathbf{v}$ and $\boldsymbol{\omega}$ respectively, and the unit vector along the upward vertical by $\boldsymbol{\gamma}$.

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The equations of motion of the sphere, relative to its principal central axes of inertia, may be written as follows:

$$m\dot{\mathbf{v}} + \boldsymbol{\omega} \times m\mathbf{v} = -mg\boldsymbol{\gamma} + \mathbf{R} \tag{3}$$

$$\boldsymbol{\theta}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{\theta}\boldsymbol{\omega} = \boldsymbol{\rho} \times \mathbf{R} \tag{4}$$

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0} \tag{5}$$

$$\mathbf{v} + \boldsymbol{\omega} \times \boldsymbol{\rho} = \mathbf{0} \tag{6}$$

Equations (3) and (4) represent the behaviour of the momentum and angular momentum of the sphere, respectively; Eqs (5) and (6) state, respectively, that the vector $\boldsymbol{\gamma}$ is constant in a fixed system of coordinates and that the sphere moves without sliding. Here $\mathbf{R}$ is the reaction of the supporting plane, $\theta = \operatorname{diag}(I_1 \;\; I_1, \;\; I_3)$ is the central inertia tensor of the sphere, and $\boldsymbol{\rho} = (-r\gamma_1, \;\; -r\gamma_2, \;\; -r\gamma_3 + a)$ is the radius-vector of the sphere's point of contact with the horizontal piane, relative to its centre of mass.

Equations (3)–(6) are closed with respect to the variables $\mathbf{v}$, $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$ and $\mathbf{R}$, but they are not in the form (1), since the system does not include a differential equation for the reaction of the plane, and Eq. (6) is not a differential equations in terms of these variables.

System (3)–(6) admits of four first integrals: the energy integral, the Jellett integral, the Chaplygin integral and a geometric integral

$$2U_0 = m\mathbf{v}^2 + I_1(\omega_1^2 + \omega_2^2) + I_3\omega_3^2 - 2mga\gamma_3 = c_0 \tag{7}$$

$$U_1 = I_1(\omega_1\gamma_1 + \omega_2\gamma_2) + I_3\omega_3(\gamma_3 - a/r) = c_1 \tag{8}$$

$$U_2 = \left[I_1 I_3 + mr^2\left(I_1(\gamma_1^2 + \gamma_2^2) + I_3(\gamma_3 - a/r)^2\right)\right]^{1/2}\omega_3 = c_2 \tag{9}$$

$$U_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{10}$$

It is clear that the expression for the total energy U_0 of the sphere involves the variable $\mathbf{v}$ which may be eliminated by using Eq. (6). When that is done U_0 becomes

$$2U_0^* = m(\boldsymbol{\omega} \times \boldsymbol{\rho})^2 + I_1(\omega_1^2 + \omega_2^2) + I_3\omega_3^2 - 2mga\gamma_3 = c_0^* \tag{11}$$

which depends, like the functions U_1, U_2 and U_3 on the variables $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ only. Note that (11) and (8)–(10) are first integrals of the equations of motion of the sphere in the form

$$\boldsymbol{\theta}\dot{\boldsymbol{\omega}} + m(\boldsymbol{\rho} \times \dot{\boldsymbol{\omega}}) \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \boldsymbol{\theta}\boldsymbol{\omega} + m[\boldsymbol{\omega} \times (\boldsymbol{\rho} \times \boldsymbol{\omega})] \times \boldsymbol{\rho} = mg\boldsymbol{\rho} \times \boldsymbol{\gamma} + m\boldsymbol{\rho} \times (\dot{\boldsymbol{\rho}} \times \boldsymbol{\omega}) \tag{12}$$

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0$$

(Eq. (12) is obtained from (4) by eliminating the reaction $\mathbf{R}$ using Eqs (3) and (6) and the equation $\dot{\mathbf{v}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} = \mathbf{0}$, which is obtained from (6) by differentiating with respect to time).

System (12), (5) is closed with respect to the variables $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$, is of the form (1), and the first integrals (11) and (8)–(10) are of the form (2). Consequently, Routh's theory can be used to investigate its steady-state solutions. When that is done the variables $\mathbf{v}$ can be found uniquely from (6), after the solutions $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ have been determined by Routh's method.

Remark. When investigating the steady motion of systems with differential constraints it is not always necessary to reduce the equations of motion to a form in which they involve neither reactions to the constraints nor undetermined multipliers. System (3)–(6) was reduced to the form (12), (5) by logical arguments. The integral (7) had to be reduced to the form of (11) in order to use Routh's theory; otherwise, formal use of the latter would have produced inaccurate results [10].

We will seek the critical points of the function U_0^* when the constants of the constant integrals (8)–(10) remain fixed. To do this we introduce the function

$$W = U_0^* - \lambda(U_1 - c_1) - \mu(U_2 - c_2) + \tfrac{1}{2}\nu(U_3 - 1)$$

where λ, μ, and ν are undetermined Lagrange multipliers. The conditions for this function to be stationary are

$$\partial W / \partial\omega_1 \equiv -mv_2(r\gamma_3 - a) + mv_3r\gamma_2 + I_1\omega_1 - I_1\lambda\gamma_1 = 0$$

$$\partial W / \partial\omega_2 \equiv mv_1(r\gamma_3 - a) - mv_3r\gamma_1 + I_1\omega_2 - I_1\lambda\gamma_2 = 0$$

$$\partial W / \partial\omega_3 \equiv -mv_1r\gamma_2 + mv_2r\gamma_1 + I_3\omega_3 - I_3\lambda(\gamma_3 - a/r) - \mu I = 0$$

$$\partial W / \partial\gamma_1 \equiv mv_2r\omega_3 - mv_3r\omega_2 - \lambda I_1\omega_1 - (\mu/I)I_1mr^2\omega_3\gamma_1 + \nu\gamma_1 = 0$$

$$\partial W / \partial\gamma_2 \equiv -mv_1r\omega_3 + mv_3r\omega_1 - \lambda I_1\omega_2 - (\mu/I)I_1mr^2\omega_3\gamma_2 + \nu\gamma_2 = 0$$

$$\partial W / \partial\gamma_3 \equiv -mga + mv_1r\omega_2 - mv_2r\omega_1 - \lambda I_3\omega_3 - (\mu/I)I_3mr^2\omega_3(\gamma_3 - a/r) + \nu\gamma_3 = 0$$

$$v_1 = (r\gamma_3 - a)\omega_2 - r\gamma_2\omega_3, \quad v_2 = r\gamma_1\omega_3 - (r\gamma_3 - a)\omega_1$$

$$v_3 = r\gamma_2\omega_1 - r\gamma_1\omega_2, \quad I = \left[I_1I_3 + mr^2\left(I_1\left(1 - \gamma_3^2\right) + I_3(\gamma_3 - a/r)^2\right)\right]^{1/2}$$

If the constants of the Jellett and Chaplygin integrals satisfy the relations

$$\left[I_1I_3 + mr^2I_3(\pm1 - a/r)^2\right]^{1/2}c_1 = I_3(\pm1 - a/r)c_2$$

then the system of equations $\delta W = 0$ will have the following respective one-parameter solutions

$$\omega_1 = \omega_2 = \gamma_1 = \gamma_2 = 0, \quad \omega_3 = \Omega = \text{const}, \quad \gamma_3 = \pm1 \tag{13}$$

(Ω is arbitrary), which represent zero-dimensional invariant sets (rest points) of system (12), (5) and correspond to permanent rotations of the sphere about a vertically located axis of dynamic symmetry at an arbitrary constant angular velocity, with the centre of mass in its lowest possible ($\gamma_3 = +1$) or highest possible ($\gamma_3 = -1$) position. In that situation it follows from (6) and (13) that $\mathbf{v} = 0$, i.e. the solutions (13) are such that the sphere's centre of mass is fixed. The undetermined Lagrange multipliers for these solutions are

$$\lambda \in R, \; \mu = I_3^{1/2}\left(I_1 + m(\pm r - a)^2\right)^{-1/2}(\Omega - \lambda(\pm1 - a/r))$$

$$\nu = \pm\left(mga + I_3\Omega[I_1\lambda + mr(\pm r - a)\Omega]\right)\left(I_1 + m(\pm r - a)^2\right)^{-1}$$

For arbitrary values of the Jellett and Chaplygin integrals, the system of equations $\delta W = 0$ will also have two-parameter families of solutions

$$\omega_1 = \omega\gamma_1, \quad \omega_2 = \omega\gamma_2, \quad \omega_3 = \omega\gamma_3 + \Omega, \quad \gamma_1^2 + \gamma_2^2 = 1 - \gamma^2, \quad \gamma_3 = \gamma \tag{14}$$

for which the three constants ω, Ω and γ satisfy one equation

$$\left[I_3\Omega+(I_3-I_1)\omega\gamma\right]\omega+mr^2(\Omega+(a/r)\omega)\omega(1-(a/r)\gamma)+mga=0$$

Formulae (14) define one-dimensional invariant sets of system (12), (5), in which the sphere has regular precessions. In that case it follows from (6) and (14) that $\nu\neq 0$. The undetermined Lagrange multipliers for the solutions (14) are

$$\lambda=\omega-(\gamma-a/r)(\Omega+a/r\omega)mr^2I_1^{-1}$$
$$\mu=\left[I_1I_3+mr^2\left(I_1\left(1-\gamma^2\right)+I_3(\gamma-a/r)^2\right)\right]^{1/2}I_1^{-1}(\Omega+(a/r)\omega)$$
$$\nu=I_1\omega^2-mr^2(\Omega+(a/r)\omega)\left[(\omega\gamma+\Omega)+\omega(\gamma-a/r)\right]$$

Note that in [10] critical points of U_0 were sought for fixed values of the constants of the integrals (8)–(10). In all the steady motions of the sphere that have been determined in [10], the centre of mass must be fixed, whereas in the general case while the regular precession occurs, the centre of mass describes a circle parallel to the horizontal plane; the centre of mass is fixed only if $\Omega+(a/r)\omega=0$, [11].

According to Routh's theory [1–9], the steady motions (13) and (14) are stable if the function U_0^* has a minimum value in these motions, when the constants of the integrals U_1, U_2 and U_3 take fixed values. This will certainly be the case if the second variation of W is positive definite on the linear manifold defined by the relations $\delta U_1=\delta U_2=\delta U_3=0$.

For the solutions (13), these relations reduce to the form $\delta\omega_3=\delta\gamma_3=0$. Then the second variation

$$2\delta^2W=A(\delta\omega_1)^2-2B(\delta\omega_1)(\delta\gamma_1)+C(\delta\gamma_1)^2+A(\delta\omega_2)^2-2B(\delta\omega_2)(\delta\gamma_2)+C(\delta\gamma_2)^2$$

is positive definite provided that $AC-B^2>0$, where

$$A=I_1+m(\pm r-a)^2>0,\quad B=I_1\lambda+mr(\pm r-a)\omega$$
$$C=\left\{I_1mr[\lambda(\pm r-a)-r\omega]\pm I_3[I_1\lambda+mr(\pm r-a)\omega]\right\}\left[I_1+m(\pm r-a)^2\right]^{-1}\omega+mr^2\omega^2\pm mga$$

Taking into account that λ was chosen arbitrarily, we conclude that a sufficient condition for the solution (13) to be stable is that at least one λ exists such that $AC-B^2>0$. Consequently, vertical rotations of the sphere are stable if

$$\left[mr(\pm r-a)\pm I_3\right]^2\omega^2\pm 4mga\left[I_1+m(\pm r-a)^2\right]>0 \tag{15}$$

Note that the sufficient condition (15) for permanent rotations of the sphere on a rough plane to be stable is identical, apart from the equality sign, with the necessary condition established in [11], whereas the sufficient condition derived in [10] from the condition that U_0 reaches a minimum for fixed values of the constants U_1, U_2 and U_3 was fairly rough and more restrictive than (15).

For the solutions (14), the second variation of W is

$$2\delta^2W=A_1(\delta\omega_1-\omega\delta\gamma_1)^2+A_2(\delta\omega_2-\omega\delta\gamma_2)^2+B_1(\delta\omega_3)^2+2B_{12}(\delta\omega_3)(\delta\gamma_3)+B_2(\delta\gamma_3)^2$$

and the linear manifold defined by $\delta U_1=\delta U_2=\delta U_3=0$ may be written as follows:

$$\alpha_1(\delta\omega_1-\omega\delta\gamma_1)+\alpha_2(\delta\omega_2-\omega\delta\gamma_2)+\alpha_3(\delta\omega_3)+\alpha_4(\delta\gamma_3)=0$$
$$\beta_3(\delta\omega_3)+\beta_4(\delta\gamma_3)=0$$
$$A_1=I_1+m(a-r\gamma)^2+mr^2\gamma_2^2>0,\quad A_2=I_1+m(a-r\gamma)^2+mr^2\gamma_1^2>0$$
$$B_1=I_3+mr^2\left(1-\gamma^2\right)>0,\quad B_{12}=-\left[I_3+mr^2\left(1-\gamma^2\right)\right]\omega+mr^2\gamma(\Omega+(a/r)\omega)$$

$$B_2 = \left[I_1 + mr^2\left(1-\gamma^2\right)\right]\omega^2 - mr^2\left(\Omega + (a/r)\omega\right)\times$$
$$\times\left\{\omega\left(\gamma - a/r\right) + \left[I_3^2 + mr^2\left[I_3\left(1-(2a/r)\gamma+\gamma^2\right) - I_1\gamma^2\right]\right]\right\}\times$$
$$\times\left[I_1I_3 + mr^2\left[I_1\left(1-\gamma^2\right) + I_3\left(\gamma - a/r\right)\right]\right]^{-1}\left(\omega\gamma+\Omega\right)\}$$
$$\alpha_1 = I_1\gamma_1,\quad \alpha_2 = I_1\gamma_2,\quad \alpha_3 = I_3\left(\gamma - a/r\right),\quad \alpha_4 = I_3\left(\omega\gamma+\Omega\right) - 2I_1\omega\gamma$$
$$\beta_3 = I_1I_3 + mr^2\left[I_1\left(1-\gamma^2\right) + I_3\left(\gamma - a/r\right)^2\right],\quad \beta_4 = mr^2\left[I_3\left(\gamma - a/r\right) - I_1\gamma\right]\left(\omega\gamma+\Omega\right)$$

The function δ^2W is positive definite on this linear manifold with respect to the variables $\delta\omega_1 - \omega\delta\gamma_1$, $\delta\omega_2 - \omega\delta\gamma_2$, $\delta\omega_3$, $\delta\gamma_3$ if the fifth- and sixth-order principal diagonal minors of the determinant

$$\Delta = \begin{vmatrix} 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 0 & \beta_3 & \beta_4 \\ \alpha_1 & 0 & A_1 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & A_2 & 0 & 0 \\ \alpha_3 & \beta_3 & 0 & 0 & B_1 & B_{12} \\ \alpha_4 & \beta_4 & 0 & 0 & B_{12} & B_2 \end{vmatrix}$$

are positive. Noting that $\Delta_5 \equiv I_1^2(A_1\gamma_2^2 + A_2\gamma_1^2)\beta_3^2 > 0$, we conclude that regular precessions of the sphere on the rough plane are stable with respect to the variables $\omega_1 - \omega\gamma_1$, $\omega_2 - \omega\gamma_2$, ω_3 and γ_3, provided that

$$\Delta_6 \equiv \Delta \equiv A_1A_2\left(\alpha_3\beta_4 - \alpha_4\beta_3\right)^2 + \left(A_1\alpha_2^2 + A_2\alpha_1^2\right)\left(B_1\beta_4^2 - 2B_{12}\beta_3\beta_4 + B_2\beta_3^2\right) > 0 \tag{16}$$

The explicit form of this condition is extremely lengthy and will therefore be omitted.

Condition (16) is identical, apart from the equality sign, with the corresponding necessary condition of [11].

In conclusion we note that a dynamically symmetric sphere is a special case of a body of revolution, whose steady motions on a rough plane have been investigated in detail in various ways [11]. The results presented above are in complete agreement with those known from previous studies, which cannot be said of the results obtained in [10].

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